q-Difference Systems Associated with Toric Varieties

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1 Introduction

As a system of differential equations corresponding to toric varieties, the GKZ system (or Mellin system of differential equations) is well known. Here, we construct a q-difference module corresponding to a compact toric variety as a semi-infinite dimensional equivariant K-cohomology, and give a q-integral representation of its solutions.

Here, a q-difference module is a module over a ring generated by a q-shift operator and operations of multiplying by functions. The q-shift operator T acts on a function $f(x)$ by $(Tf)(x) = f(qx)$. In the context of mirror symmetry, the GKZ system is known to compute the quantum cohomology of a toric variety. There is a possibility that the q-difference module given here is related to quantum K-cohomology.

2 Toric Varieties

Let us construct a toric variety X from the following data.

i) An r-dimensional torus $\mathbb{T} \cong (\mathbb{C}^*)^r$

ii) N integral weight vectors $u_1, \dots, u_N \in \mathrm{Hom}(\mathbb{T}, \mathbb{C}^*) \cong \mathbb{Z}^r$ (allowing multiplicities)

iii) An element specifying the Kähler class $\eta \in \sum_{i=1}^N \mathbb{R}_{\geq 0} u_i \subset \mathrm{Hom}(\mathbb{T}, \mathbb{C}^*) \otimes \mathbb{R}$

For the above data, we define a subset $\mathcal{A}$ of the power set of $\{1, \dots, N\}$ by $\mathcal{A} = \{I \subset \{1, \dots, N\}; \eta \in \sum_{i \in I} \mathbb{R}_{\geq 0} u_i\}$, and define an open set $\mathcal{U}_\eta \subset \mathbb{C}^N$ determined by η as

$$\mathcal{U}_\eta = \mathbb{C}^N \setminus \bigcup_{I \notin \mathcal{A}} \mathbb{C}^I = \bigcup_{I \in \mathcal{A}} \mathbb{C}^{*I} \times \mathbb{C}^{\overline{I}}$$

Here, $\mathbb{C}^I = \{(z_1, \dots, z_N) \in \mathbb{C}^N; z_i = 0 \text{ for } i \notin I\}$, and $\overline{I} \subset \{1, \dots, N\}$ is the complement of I. The torus $\mathbb{T}$ acts on $\mathbb{C}^N$ component-wise by the weights $u_1, \dots, u_N$, and in particular acts on the open set $\mathcal{U}_\eta$. The toric variety X is defined as the quotient of $\mathcal{U}_\eta$ by $\mathbb{T}$:

$$X = \mathcal{U}_\eta / \mathbb{T}$$

This can also be written as a symplectic quotient by the maximal compact subgroup $\mathbb{T}_\mathbb{R} \cong (S^1)^r$ of $\mathbb{T}$:

$$X \cong \left\{ (z_1, \dots, z_N) \in \mathbb{C}^N; \textstyle\sum_{i=1}^N |z_i|^2 u_i = \eta \right\} \Big/ \mathbb{T}_\mathbb{R}$$

The toric variety given here is complete with respect to the reduced Kähler metric, and in the language of fans, the support of the fan is convex[1]. X is an orbifold when it satisfies condition (a) below, and is smooth if it further satisfies (b). It becomes compact if it satisfies (c).

[1] For example, the total space of $\mathcal{O}(-1)$ on $\mathbb{P}^1$ is such a case, but the total space of $\mathcal{O}(1)$ ($= \mathbb{P}^2 \setminus \{\mathrm{pt}\}$) is not.

(a) For each $I \in \mathcal{A}$, $\{u_i\}_{i\in I}$ spans the vector space Hom$(\mathbb{T}, \mathbb{C}^*) \otimes \mathbb{R}$.

(b) For each $I \in \mathcal{A}$, $\{u_i\}_{i\in I}$ generates the lattice Hom$(\mathbb{T}, \mathbb{C}^*)$.

(c) $\{(c_1, \ldots, c_N) \in \mathbb{R}^N; c_i \geq 0, \sum_{i=1}^N c_i u_i = 0\} = \{0\}$.

In the following, we assume conditions (b) and (c), and let X be smooth and compact.

Elements of the weight lattice Hom$(\mathbb{T}, \mathbb{C}^*)$ determine line bundles on $X = \mathcal{U}_\eta/\mathbb{T}$, and by assigning their Chern classes, a natural isomorphism Hom$(\mathbb{T}, \mathbb{C}^*) \cong H^2(X, \mathbb{Z})$ is obtained. In this case, the Kähler cone consisting of classes represented by Kähler forms in $H^2(X, \mathbb{R})$ is written as $\bigcap_{I\in\mathcal{A}}(\sum_{i\in I} \mathbb{R}_{>0} u_i)$. (Under assumption (a), η is an interior point of this cone.) Let the coordinates of $\mathbb{C}^N$ be $z_1, \ldots, z_N$, and let the divisor in X defined by $z_i = 0$ be $D_i = \{z_i = 0\} \cap \mathcal{U}_\eta/\mathbb{T}$.

Proposition 2.1. The K-cohomology ring (of holomorphic vector bundles) of the toric variety X is generated as a ring over $\mathbb{Z}$ by the classes of the line bundles $\mathcal{O}(D_i)^{\pm}$. Let $U_i := [\mathcal{O}(D_i)]$ be the class in the K-group; then all relations are generated by the following two types.

(i) $\prod_{i=1}^N U_i^{c_i} = 1$ for $\sum_{i=1}^N c_i u_i = 0, c_i \in \mathbb{Z}$,

(ii) $\prod_{i\in \bar{I}}(1 - U_i^{-1}) = 0$ for $I \notin \mathcal{A}$.

For later purposes, we fix a $\mathbb{Z}$-basis $p_1, \ldots, p_r$ of $H^2(X, \mathbb{Z})$ that is contained in the closure of the Kähler cone. Let the classes in the K-group of the corresponding line bundles be $P_1, \ldots, P_r$. (Pic$(X) \cong H^2(X, \mathbb{Z})$.) In this case, setting $u_i = \sum_{a=1}^r u_i^a p_a$, we have

$$U_i = \prod_a P_a^{u_i^a}$$

and $d \in H_2(X, \mathbb{Z})$ is represented in coordinates as $d = (d_1, \ldots, d_r),\ d_a = \langle p_a, d\rangle$. If d is a class representing a curve, the choice of p_a implies $d_a \geq 0$.

3 Loop Space Model and Semi-Infinite Dimensional K-theory

As a model of the universal cover of the loop space of the toric variety, we consider the following L_X:

$$L_X = L\mathcal{U}_\eta/\mathbb{T}, \quad L\mathcal{U}_\eta = \mathbb{C}[\zeta, \zeta^{-1}]^N \setminus \bigcup_{I\notin\mathcal{A}} \mathbb{C}[\zeta, \zeta^{-1}]^I$$

Here, $\mathbb{T}$ acts on the infinite-dimensional vector space $\mathbb{C}[\zeta, \zeta^{-1}]^N$ by the weights $u_1, \ldots, u_N$ componentwise as before. The variable ζ is considered the loop parameter. L_X is an infinite-dimensional toric variety, and in terms of the data from the previous section, it corresponds to keeping the torus $\mathbb{T}$ of i) and the Kähler class η of iii) unchanged, while for the vectors in ii), we prepare countably infinite copies for each of u_1 to u_N. L_X has an S^1-action by loop rotation $\zeta \mapsto e^{\sqrt{-1}\theta}\zeta$. This S^1-action is Hamiltonian, and the Hamiltonian function $H\colon L_X \to \mathbb{R}$ can be written in the form:

$$H[\gamma_1(\zeta), \ldots, \gamma_N(\zeta)] = \sum_{i=1}^N \sum_\nu \nu |z_{i\nu}|^2$$

where $\gamma_i(\zeta) = \sum_\nu z_{i\nu}\zeta^\nu \in \mathbb{C}[\zeta, \zeta^{-1}]$, and $z_{i\nu}$ are normalized to satisfy $\sum_{i,\nu} |z_{i\nu}|^2 u_i = \eta$ so that they lie in the level set of the symplectic reduction. In the following, we want to consider Morse theory on L_X with respect to the Hamiltonian H. The downward Morse flow ϕ_t for the above H is written as

$$\phi_t[\gamma(\zeta)] = [\gamma(e^{-t}\zeta)], \quad \gamma(\zeta) \in L\mathcal{U}_\eta$$

(normalization is not imposed here). Each component of the critical manifold of H is isomorphic to X and is parameterized by $d \in H_2(X, \mathbb{Z}) \cong \mathrm{Hom}(\mathbb{T}, \mathbb{C}^*)^\vee$:

$$X_d = \{[z_1\zeta^{\langle u_1, d\rangle}, \dots, z_N\zeta^{\langle u_N, d\rangle}] \in L_X\} \cong X$$

The stable manifold $\widetilde{L}_d^\infty$ and unstable manifold $\widetilde{L}_{-\infty}^d$ of X_d with respect to the Morse flow ϕ_t are given by the following:

$$\widetilde{L}_d^\infty = \left\{[\gamma_1(\zeta), \dots, \gamma_N(\zeta)] \in L_X; \gamma_i(\zeta) = \textstyle\sum_{\nu \geq \langle u_i, d\rangle} z_{i\nu}\zeta^\nu, (z_{1\langle u_1, d\rangle}, \dots, z_{N\langle u_N, d\rangle}) \in \mathcal{U}_\eta\right\}$$
$$\widetilde{L}_\infty^d = \left\{[\gamma_1(\zeta), \dots, \gamma_N(\zeta)] \in L_X; \gamma_i(\zeta) = \textstyle\sum_{\nu \leq \langle u_i, d\rangle} z_{i\nu}\zeta^\nu, (z_{1\langle u_1, d\rangle}, \dots, z_{N\langle u_N, d\rangle}) \in \mathcal{U}_\eta\right\}$$

We denote their closures by L_d^∞ and $L_{-\infty}^d$ (removing the second condition). Since the original toric variety X is compact, we have

$$L_X = \bigsqcup_d \widetilde{L}_d^\infty = \bigsqcup_d \widetilde{L}_{-\infty}^d = \bigsqcup_{d_1, d_2} (\widetilde{L}_{d_1}^\infty \cap \widetilde{L}_{-\infty}^{d_2})$$

and it turns out that $L_{d_1}^\infty \cap L_{-\infty}^{d_2}$ is a compact smooth toric variety. Additionally, L_X, L_d^∞, and $L_{-\infty}^d$ give the classifying space $B\mathbb{T}$ of $\mathbb{T}$ homotopy-theoretically.

Using the above S^1-equivariant stratification, we construct the semi-infinite equivariant K-cohomology as follows. First, for $d, d' \in H_2(X, \mathbb{Z})$, we define an order relation $d \prec d'$ by the inclusion relation $L_d^\infty \supset L_{d'}^\infty$. This is equivalent to $\langle u_i, d\rangle \leq \langle u_i, d'\rangle, \forall i$. When $d_1 \prec d_2$, $L_{d_1}^\infty \cap L_{-\infty}^{d_2}$ is a compact smooth toric variety, and for $d_1' \prec d_1 \prec d_2$, a push-forward (Gysin map) $K_{S^1}(L_{d_1}^\infty \cap L_{-\infty}^{d_2}) \to K_{S^1}(L_{d_1'}^\infty \cap L_{-\infty}^{d_2})$ is defined. For any d_2, a map $K_{S^1}(L_{d_1}^\infty) \to K_{S^1}(L_{d_1'}^\infty)$ making the following diagram commutative is uniquely determined:

$$\begin{array}{ccc} K_{S^1}(L_{d_1}^\infty) & \longrightarrow & K_{S^1}(L_{d_1'}^\infty) \\ \downarrow & & \downarrow \\ K_{S^1}(L_{d_1}^\infty \cap L_{-\infty}^{d_2}) & \longrightarrow & K_{S^1}(L_{d_1'}^\infty \cap L_{-\infty}^{d_2}) \end{array}$$

Here the vertical maps are restrictions. By this, $K_{S^1}(L_d^\infty)$ forms a directed system, and we set its limit as

$$K_{S^1}^{\infty/2}(L_X) := \mathrm{inj}\lim_d K_{S^1}(L_d^\infty), \quad \text{where } d \text{ goes in the decreasing direction}$$

Similarly, for $L_{-\infty}^d$, $K_{S^1}(L_{-\infty}^d)$ forms a directed system, and the dual theory is defined by

$$K_{\infty/2}^{S^1}(L_X) := \mathrm{inj}\lim_d K^{S^1}(L_{-\infty}^d), \quad \text{where } d \text{ goes in the increasing direction}$$

When $d_1 \prec d_2$, the equivariant Euler characteristic $\chi_{S^1}\colon K_{S^1}(L_{d_1}^\infty \cap L_{-\infty}^{d_2}) \to \mathbb{Z}[q, q^{-1}]$ is defined. Here, for a vector bundle E, $\chi_{S^1}(E) = \sum_i (-1)^i \,\mathrm{Ch}[H^i(X, E)]$, where $\mathrm{Ch}(\cdot)$ is the character of the S^1 representation, and q is the variable corresponding to the fundamental representation of S^1. This induces the following pairing between the direct limits:

$$\chi_{S^1}\colon K_{\infty/2}^{S^1}(L_X) \times K_{S^1}^{\infty/2}(L_X) \to \mathbb{Z}[q, q^{-1}]$$

To concretely describe the semi-infinite dimensional K-theory, we introduce the S^1-equivariant line bundles $\widehat{P}_a$ and $\widehat{U}_{i\nu}$ on L_X as follows. (Their classes in the K-group are represented by the same symbols.)

$$\widehat{P}_a = L\mathcal{U}_\eta \times \mathbb{C}/(\gamma, v) \sim (t\gamma, p_a(t)v), t \in \mathbb{T} \quad S^1 \text{ action: } [\gamma(\zeta), v] \mapsto [\gamma(e^{\sqrt{-1}\theta}\zeta), v]$$
$$\widehat{U}_{i\nu} = L\mathcal{U}_\eta \times \mathbb{C}/(\gamma, v) \sim (t\gamma, u_i(t)v), t \in \mathbb{T} \quad S^1 \text{ action: } [\gamma(\zeta), v] \mapsto [\gamma(e^{\sqrt{-1}\theta}\zeta), e^{\nu\sqrt{-1}\theta}v]$$

Then the following holds:

$$\widehat{U}_{i\nu} = q^{\nu} \prod_{a=1}^{r} \widehat{P}_a^{u_i^a}, \quad K_{S^1}(L_d^{\infty}) \cong K_{S_1}(L_{-\infty}^{d}) \cong \mathbb{Z}[P_1^{\pm}, \dots, P_r^{\pm}, q^{\pm}].$$[2]

Also, the push-forward map $K_{S^1}(L_d^{\infty}) \to K_{S^1}(L_{d'}^{\infty})$ is given by multiplication by the Thom class

$$\alpha \mapsto \alpha \cdot \prod_{i=1}^{N} \prod_{\nu=\langle u_i, d'\rangle}^{\langle u_i, d\rangle - 1} (1 - \widehat{U}_{i\nu}^{-1})$$

Let $\Delta \in K_{S^1}^{\infty/2}(L_X)$ be the image of $1 \in K_{S^1}(L_0^{\infty})$. Pushing 1 forward along the sequence of push-forwards

$$K_{S^1}(L_0^{\infty}) \to K_{S^1}(L_{d_1}^{\infty}) \to K_{S^1}(L_{d_2}^{\infty}) \to \cdots, \quad 0 \succ d_1 \succ d_2 \succ \cdots$$

we obtain the following infinite product representation of the element Δ:

$$\Delta = \prod_{i=1}^{N} \prod_{\nu < 0} (1 - \widehat{U}_{i\nu}^{-1})$$

Let us endow the semi-infinite dimensional K-groups with a q-difference module structure. L_X was a model of the universal cover of the loop space, and for each $d \in H_2(X, \mathbb{Z})$, a map $Q^d \colon L_X \to L_X$ corresponding to a covering transformation is defined:

$$Q^d[\gamma_1(\zeta), \dots, \gamma_N(\zeta)] = [\zeta^{-\langle u_1, d\rangle}\gamma_1(\zeta), \dots, \zeta^{-\langle u_N, d\rangle}\gamma_N(\zeta)]$$

Letting $Q_1, \dots, Q_r$ be the covering transformations corresponding to the basis of $H_2(X, \mathbb{Z})$ dual to the basis $p_1, \dots, p_r$, Q^d is represented as $Q^d = Q_1^{d_1} \cdots Q_r^{d_r}$. This induces a pull-back $Q^d \colon K_{S^1}(L_{d'}^{\infty}) \to K_{S^1}(L_{d'+d}^{\infty})$ and a push-forward $Q^d \colon K_{S^1}(L_{-\infty}^{d'}) \to K_{S^1}(L_{-\infty}^{d'-d})$, and passing to the direct limit determines:

$$Q^d \colon K_{S^1}^{\infty/2}(L_X) \to K_{S^1}^{\infty/2}(L_X), \quad Q^d \colon K_{\infty/2}^{S^1}(L_X) \to K_{\infty/2}^{S^1}(L_X)$$

On the other hand, $\widehat{P}_1, \dots, \widehat{P}_r$ act on $K_{S^1}^{\infty/2}(L_X)$ and $K_{\infty/2}^{S^1}(L_X)$ by multiplication, and the actions of $\widehat{P}_a$ and Q_b satisfy the following:

$$Q_b \widehat{P}_a = q^{\delta_{ab}} \widehat{P}_a Q_b \quad \text{on } K_{S^1}^{\infty/2}(L_X),$$
$$Q_b \widehat{P}_a = q^{-\delta_{ab}} \widehat{P}_a Q_b \quad \text{on } K_{\infty/2}^{S^1}(L_X).$$

Therefore, on $K_{S^1}^{\infty/2}(L_X)$, $\widehat{P}_a$ acts the same as the shift operator $Q_b \mapsto q^{-\delta_{ab}} Q_b$, and $K_{S^1}^{\infty/2}(L_X)$ has the structure of a q-difference module. The automorphism of L_X reversing the direction of the loop $[\gamma(\zeta)] \mapsto [\gamma(\zeta^{-1})]$ (which reverses the S^1-action) induces an isomorphism

$$\overline{} \colon K_{S^1}^{\infty/2}(L_X) \xrightarrow{\cong} K_{\infty/2}^{S^1}(L_X)$$

This map satisfies $\overline{Q_a \alpha} = Q_a \overline{\alpha}$, $\overline{\widehat{P}_a \alpha} = \widehat{P}_a \overline{\alpha}$, and $\overline{q\alpha} = q^{-1}\overline{\alpha}$.

Proposition 3.1. *$K_{S^1}^{\infty/2}(L_X)$ is generated by Δ as a module over the ring of difference operators $\mathbb{Z}\langle P_1^{\pm}, \dots, P_r^{\pm}, Q_1^{\pm}, \dots, Q_r^{\pm}, q^{\pm}\rangle$, and all relations are generated by the following, where $d \in H_2(X, \mathbb{Z})$:*

$$\left[Q^d \prod_{\langle u_i, d\rangle < 0} \prod_{\nu=0}^{-\langle u_i, d\rangle - 1} \left(1 - q^{-\nu} \textstyle\prod_a \widehat{P}_a^{-u_i^a}\right) - \prod_{\langle u_i, d\rangle > 0} \prod_{\nu=0}^{\langle u_i, d\rangle - 1} \left(1 - q^{-\nu} \textstyle\prod_a \widehat{P}_a^{-u_i^a}\right) \right] \Delta = 0$$

[2] In fact, it is an appropriate completion of this, but for simplicity, we write it this way here.

Next, we construct a power series solution of $K^{\infty/2}_{S^1}(L_X)$. For $\alpha \in K^{\infty/2}_{S^1}(L_X)$, there exists some $d \prec 0$ such that α is represented by some element $\alpha_d \in K_{S^1}(L^\infty_d)$. On the other hand, $\Delta \in K^{\infty/2}_{S^1}(L_X)$ is represented by the element $\prod_i \prod_{\nu=\langle u_i,d\rangle}^{-1}(1-\widehat{U}_{i\nu}^{-1})$ in $K_{S^1}(L^\infty_d)$. Thus, for the embedding $i\colon X \cong X_0 \to L_X$, we define

$$i^*\left(\frac{\alpha}{\Delta}\right) := i^*\left(\frac{\alpha_d}{\prod_{i=1}^N \prod_{\nu=\langle u_i,d\rangle}^{-1}(1-\widehat{U}_{i\nu}^{-1})}\right) = \frac{i^*(\alpha_d)}{\prod_{i=1}^N \prod_{\nu=\langle u_i,d\rangle}^{-1}(1-q^{-\nu}\prod_a P_a^{-u_i^a})}$$

The right-hand side is independent of the choice of d, and determines an element of $K(X) \otimes \mathbb{Z}[q^\pm,(1-q^n)^{-1}; n>0]$. We define a map Loc: $K^{\infty/2}_{S^1}(L_X) \to K(X)\otimes \mathbb{Z}[q^\pm,(1-q^n);n>0][Q^{-1},Q]\!]$ by[3]

$$\mathsf{Loc}(\alpha) = \sum_{d\in H_2(X,\mathbb{Z})} i^*\left(\frac{Q^{-d}\alpha}{\Delta}\right)Q^d$$

Let T_a be the shift operator defined by $T_a(Q_b) = q^{\delta_{ab}}Q_b$.

Proposition 3.2. *The map* Loc *is a homomorphism of* $\mathbb{Z}[Q_1^\pm,\dots,Q_r^\pm]$*-modules and satisfies the difference equation*

$$P_a T_a^{-1}\,\mathsf{Loc}(\alpha) = \mathsf{Loc}(\widehat{P}_a\alpha)$$

Furthermore, Loc *is injective.*

Corollary 3.3. The function $I(Q,q) := \prod_a P_a^{-\log Q_a/\log q}\,\mathsf{Loc}(\Delta)$ taking values in $K(X)$ is a (generally formal) power series solution of the following difference equation:

$$\left[Q^d \prod_{\langle u_i,d\rangle<0}\prod_{\nu=0}^{-\langle u_i,d\rangle-1}\left(1-q^{-\nu}\prod_a T_a^{u_i^a}\right) - \prod_{\langle u_i,d\rangle>0}\prod_{\nu=0}^{\langle u_i,d\rangle-1}\left(1-q^{-\nu}\prod_a T_a^{u_i^a}\right)\right] I(Q,q) = 0$$

where $P_a^z = (1+(P_a-1))^z = 1+z(P_a-1)+\binom{z}{2}(P_a-1)^2+\cdots$, $d\in H_2(X,\mathbb{Z})$. (Note that P_a-1 is nilpotent in the K-group.)

The solution $I(Q,q)$ is concretely given by:

$$I(Q,q) = \prod_a P_a^{-\log Q_a/\log q}\sum_{d\geq 0}\prod_{i=1}^N \frac{\prod_{m=-\infty}^0(1-q^m\prod_a P_a^{-u_i^a})}{\prod_{m=-\infty}^{\langle u_i,d\rangle}(1-q^m\prod_a P_a^{-u_i^a})}Q^d$$

4 Pairing of q-Difference Modules and Completion

The pairing χ_{S^1} between $K^{S^1}_{\infty/2}(L_X)$ and $K^{\infty/2}_{S^1}(L_X)$ extends to a $\mathbb{Z}[Q^\pm]$-bilinear pairing $G(\cdot,\cdot)$:

$$G(\alpha,\beta) := \sum_{d\in H_2(X,\mathbb{Z})} \chi_{S^1}(\alpha, Q^{-d}\beta)Q^d, \quad \alpha\in K^{S^1}_{\infty/2}(L_X), \beta\in K^{\infty/2}_{S^1}(L_X).$$

This takes values in $\mathbb{Z}[q,q^{-1}][Q^{-1},Q]\!]$. By the localization theorem in equivariant K-theory, we obtain the following.

Proposition 4.1. *For* $\alpha,\beta\in K^{\infty/2}_{S^1}(L_X)$*, the following holds:*

$$G(\overline{\alpha},\beta) = \chi(\overline{\mathsf{Loc}(\alpha)}\otimes \mathsf{Loc}(\beta))$$

Here, χ *is the Euler characteristic in* $K(X)$*, and* $\overline{}$ *is defined on* $K(X)\otimes\mathbb{Z}[q^\pm,(1-q^n)^{-1};n>0][Q^{-1},Q]\!]$ *such that* $\overline{q}=q^{-1}$.

[3] Loc stands for localization.

We define a submodule $F^n(K_{S^1}^{\infty/2}(L_X))$ of $K_{S^1}^{\infty/2}(L_X)$ as follows:

$$F^n(K_{S^1}^{\infty/2}(L_X)) := \sum_{\substack{i_1+\cdots+i_r\geq n,\\ i_1,\dots,i_r\geq 0}} Q_1^{i_1}\cdots Q_r^{i_r}\mathbb{Z}\langle P_1^{\pm},\dots,P_r^{\pm},Q_1,\dots,Q_r,q^{\pm}\rangle\Delta$$

This filtration defines a topology on $F^0(K_{S^1}^{\infty/2}(L_X))$. We define $FK_{S^1}(L_X)$ as the completion with respect to this topology[4]:

$$FK_{S^1}(L_X) := \widehat{F^0}(K_{S^1}^{\infty/2}(L_X)) = \operatorname{proj}\lim_n F^0(K_{S^1}^{\infty/2}(L_X))/F^n(K_{S^1}^{\infty/2}(L_X))$$

$FK_{S^1}(L_X)$ becomes a module over the completed difference operator ring $\mathbb{Z}[q^{\pm},P_1^{\pm},\dots,P_r^{\pm}][\![Q_1,\dots,Q_r]\!]$. The filtration on $K^{S^1}_{\infty/2}(L_X)$ and its corresponding completion $FK^{S^1}(L_X)$ are defined similarly. The map Loc, the pairing $G(\cdot,\cdot)$, and the isomorphism $\overline{}$ are extended to the completed modules as follows, satisfying Propositions 3.2 and 4.1:

$$\begin{aligned}\mathrm{Loc}\colon FK_{S^1}(L_X) &\hookrightarrow K(X)\otimes\mathbb{Z}[q^{\pm},(1-q^n)^{-1};n>0][\![Q]\!]\\ G(\cdot,\cdot)\colon FK^{S^1}(L_X)\otimes_{\mathbb{Z}[q,q^{-1}][\![Q]\!]}FK_{S^1}(L_X) &\to \mathbb{Z}[q,q^{-1}][\![Q]\!]\\ \overline{}\colon FK_{S^1}(L_X) &\xrightarrow{\cong} FK^{S^1}(L_X)\end{aligned}$$

Proposition 4.2. *$FK_{S^1}(L_X)$ is free as a module over $\mathbb{Z}[q,q^{-1}][\![Q]\!]$, and has a canonical isomorphism:*

$$FK_{S^1}(L_X)\Big/\sum_{a=1}^r Q_aFK_{S^1}(L_X)\cong K(X)\otimes\mathbb{Z}[q,q^{-1}]$$

Also, the pairing $G(\overline{\alpha},\beta)$ on $FK_{S^1}(L_X)$ is non-degenerate and satisfies $G(\overline{\alpha},\beta)|_{Q=0}=\chi(\overline{\alpha}_0\otimes\beta_0)$. Here α_0,β_0 are elements in $K(X)\otimes\mathbb{Z}[q,q^{-1}]$ corresponding to α,β under the above isomorphism.

In quantum K-theory, it is conjectured that a q-difference module structure is determined on $K(X)\otimes\mathbb{Z}[q,q^{-1}][\![Q]\!]$, and the author conjectures that there is some relation between the above $FK_{S^1}(L_X)$ and quantum K-theory. However, in the above Proposition, it is only known that $FK_{S^1}(L_X)$ is a free module of rank equal to dim $K(X)$, and the problem is that a method to canonically identify it with $K(X)\otimes\mathbb{Z}[q,q^{-1}][\![Q]\!]$ is not given. In this conference, we also mentioned a certain method for constructing quantum K-theory starting from q-difference modules. According to this, a certain conjecture about the relationship between $FK_{S^1}(L_X)$ and quantum K-theory can be formulated, which is planned to be stated in a paper currently in preparation.

5 q-Integral Representation of Solutions

Here we give a q-integral representation of the solutions to the difference module $K_{S^1}^{\infty/2}(L_X)$. What is given below is a q-analog of the oscillatory integrals usually given as toric mirrors. Consider the family of complex tori dual to the embedding $\mathbb{T}\hookrightarrow(\mathbb{C}^*)^N$ determined by the weights $u_1,\dots,u_N$:

$$\pi\colon Y=(\mathbb{C}^*)^N\to\mathbb{T}^\vee,\quad (x_1,\dots,x_N)\mapsto(Q_1,\dots,Q_r),\ Q_a=\prod_{i=1}^N x_i^{u_i^a}$$

We call this the mirror family of X. Each fiber $Y_Q:=\pi^{-1}(Q_1,\dots,Q_r)$ is isomorphic to the $(N-r)(=\dim X)$-dimensional torus $\operatorname{Ker}\pi=Y_1\cong(\mathbb{C}^*)^{N-r}$. For a function $f(y)$ on Y_Q and $a\in Y_Q$, the q-integral of $f(y)$ is defined as

$$\int_{Y_Q}^a f(y)d_qy := (1-q)\sum_{v\in\mathrm{Hom}(\mathbb{C}^*,Y_1)} f(v(q)\cdot a)$$

[4] FK stands for Floer K-theory.

Choose a splitting homomorphism $s\colon \mathbb{T}^\vee \to Y$ of the surjection π and $\beta \in Y_1$, and consider the following q-integral $\mathcal{I}_{s,\beta}(Q,q)$:

$$\mathcal{I}_{s,\beta}(Q,q) = \int_{Y_Q}^{\beta\cdot s(Q)} \prod_{i=1}^{N} \mathrm{exp}_q\left(\frac{x_i}{1-q}\right) d_q y$$

Here, $x_1, \ldots, x_N$ are the coordinates of Y, and the q-exponential function $\mathrm{exp}_q(x)$ is defined by

$$\mathrm{exp}_q(x) := \sum_{n=0}^{\infty} \frac{(1-q)^n}{(1-q)\cdots(1-q^n)} x^n.$$

$\mathrm{exp}_q(x/(1-q))$ appearing in the integrand has no zeros when $|q|<1$, and has simple poles at $x = q^{-n}, n \geq 0$. Also, for $|q|<1$, it is expanded into the following infinite product:

$$\mathrm{exp}_q\left(\frac{x}{1-q}\right) = \frac{1}{\prod_{n=0}^{\infty}(1-q^n x)} = \frac{1}{(x;q)_\infty}$$

Proposition 5.1. Assume that the q-integral $\mathcal{I}_{s,\beta}(Q,q)$ converges. Then, $\mathcal{I}_{s,\beta}(Q,q)$ provides a solution to the difference equation in Corollary 3.3.

Example 5.2. *For $X = \mathbb{P}^1$, the family of tori $\pi\colon Y = (\mathbb{C}^*)^2 \ni (x_1,x_2) \mapsto Q = x_1x_2 \in \mathbb{C}^*$ is the corresponding mirror. When taking the splitting $Q \mapsto (1,Q)$, the corresponding q-integral solution is given by*

$$\mathcal{I}_\beta(Q,q) = \int_{\mathbb{C}^*}^{\beta} \frac{d_q y}{(y;q)_\infty (Q/y;q)_\infty} = \sum_{n\in\mathbb{Z}} \frac{1-q}{(\beta q^n;q)_\infty (\frac{Q}{\beta q^n};q)_\infty}$$

and it satisfies the difference equation $(1-T)^2\mathcal{I}_\beta(Q,q) = Q\mathcal{I}_\beta(Q,q)$.

References are given below. The description of quantum cohomology and its mirror for toric varieties and complete intersections within them was given in [2]. The description of quantum cohomology as semi-infinite dimensional cohomology is found in [1, 5]. As foundational literature on quantum K-theory, there is [4], which showed the relationship between the quantum K-theory of type A flag manifolds and difference Toda lattices, and [3, 6], which describe structures close to Frobenius manifolds (structure of an F-manifold + affine structure + flat metric).

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